JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 30, 596–604 (1970)

The Spectral Distribution for a Differential Equation Associated with infrasonic Waves

VAUGHAN H. WESTON

Radiation Laboratory, University of Michigan, Ann Arbor, Michigan 48104

Submitted by C. L. Dolph

1. INTRODUCTION

The spectrum of the differential equation

$$\frac{d}{dz}\left[p\frac{dy}{dz}\right]+(\lambda-q)y=0 \tag{1}$$

is considered for the interval $(0, \infty)$, for the case where the boundary conditions at $z=0$ and $z=\infty$ are limit circle and limit point, respectively, and there exists a singularity at a point in the interval. This singularity is characterized by the existence of a pole of order one belonging to $p(z)$. It can be shown that at this singular point, both independent solutions are finite, whereas their derivatives vanish.

An equation of the form (1), with the associated singularity in $p(z)$, occurs in the study of time harmonic infrasonic waves [1], where the coefficients p and q are functions of frequency and vertical temperature profile, and y is proportioned to the excess pressure. The problem wherein p possesses a pole occurs at low frequencies, and the modes corresponding to the discrete spectrum are important in the study of long distance propagation of the pressure pulse produced by large explosions in the atmosphere. The existence and number of the discrete modes depend upon the structure of the atmosphere, especially upon the functional dependence of the ambient temperature on altitude [2].

For the usual case where $p(z)$ does not possess a pole, a suitable transformation upon both y and z places Eq. (1) in the normal form

$$\frac{d^2y}{dx^2}+y(\lambda-Q(x))=0, \tag{2}$$

where $Q(x)$ is $L(0, \infty)$. It is shown by Titchmarsh [3] for this case, that there is a continuous spectrum in $0 \leqslant \lambda \leqslant \infty$, and that there may exist a discrete spectrum in $(-\infty < \lambda < 0)$ which is bounded below ([3], [2]). In contrast to this, it will be shown here, that for a simple pole, the discrete spectrum is unbounded in $-\infty \leqslant \lambda < 0$. There will be no change in the continuous spectrum, since the presence of the pole at $z = z_0$ does not produce any change in the integrability properties of independent solutions in the neighborhood of $z = z_0$. Because of this, only the discrete spectrum will be investigated. A general treatment on the existence of the discrete spectrum, followed by an asymptotic development on the location of the discrete spectrum for $\lambda \to -\infty$, will be given.

The functional behaviour of $p(z)$ and $q(z)$ will be unspecified apart from the following conditions. $p(z)$ will be related to the twice differentiable function $h(z)$

$$p(z) = (z - z_0)^{-1} h(z), \tag{i}$$

where $h(z)$ is nonzero in the range $0 \leqslant z < \infty$; and $h(z)$ will have the following behavior for $z \to \infty$:

$$h(z) \sim 0(z^{\alpha}), \qquad \alpha < \tfrac{5}{3}. \tag{ii}$$

In the analysis that will be given in this paper, an additional restriction upon $h(z)$ will be required, namely that

$$h'(z_0) = 0. \tag{iii}$$

For simplification, $p(z)$ will be normalized so $p(0) = -1$. This does not place any further restriction on $p(z)$. The function $q(z)$ must satisfy the following integrability conditions

$$\int_0^z q(z)\, dz < \infty \tag{iv}$$

for $0 \leqslant z < \infty$, and

$$\int_z^\infty p^{1/2}(z)\, q(z)\, dz < \infty \tag{v}$$

for $z_0 < z \leqslant \infty$.

2. The Associated Differential Equation

As a first step in the analysis, the appropriate associated differential equation having the following form

$$\frac{d}{dz}\left[p \frac{dy}{dz}\right] + (\lambda - q_1)\, y = 0, \tag{3}$$

where $q_1(z)$ belongs to $L(0, \infty)$, will be sought, for which exact solutions are known. Such an equation may be found by reducing Eq. (1), by the substitution $y = p^{-1/2}w$, to an equation of the general form

$$\frac{d^2w}{dz^2} + w\left[\lambda(z - z_0)\,\psi_1 - \frac{\frac{3}{4}}{(z - z_0)^2} + \chi\right] = 0,$$

where $\chi(z)$ and $\psi_1(z)$ are bounded functions independent of λ. It is seen [4] that the asymptotic solution of the above equation for $|\lambda| \to \infty$ displays Stoke's phenomenon. Hence, it is suggested by the analysis in [4] that the appropriate associated equation should be

$$\frac{d^2w}{dz^2} + w\left[s^2\phi^2 - \frac{1}{3}\frac{\phi^2}{\Phi^2} - \frac{\Psi''}{\Psi}\right] = 0, \tag{4}$$

where

$$s^2 = \lambda, \qquad 0 \leqslant \arg s < \pi \tag{5}$$

$$\phi^2 = p^{-1} \tag{6}$$

$$\Phi = \int_{z_0}^{z} \phi\, dz \tag{7}$$

$$\Psi = \Phi^{1/6}\phi^{-1/2}, \tag{8}$$

since $\frac{1}{3}\phi^2\Phi^{-2}$ differs from $3/[4(z - z_0)^2]$ only by an analytic function. Because ϕ^2 has a zero of order one at $z = z_0$, the appropriate Riemann sheet will have to be specified in the definition of ϕ. In this connection, the following arguments of ϕ and Φ will be prescribed

$$\arg\phi = 0, \qquad \arg\Phi = 0 \qquad \text{for} \qquad z > z_0;$$

$$\arg\phi = \frac{\pi}{2}, \qquad \arg\Phi = \frac{3\pi}{2} \qquad \text{for} \qquad z < z_0.$$

The general solution of Eq. (4) is given by

$$w = \xi^{1/2}\phi^{-1/2}C_{2/3}(\xi),$$

where

$$\xi = s\Phi \tag{8}$$

and the functions $C_{2/3}(\xi)$ are Bessel functions of order $\frac{2}{3}$.

Transforming Eq. (4) back into an equation of the form of (3), it can be shown that

$$q_1(z) = \Psi\frac{d}{dz}\left(\frac{1}{\phi^2}\frac{d\Psi^{-1}}{dz}\right). \tag{9}$$

Thus, with the choice of q_1 given by Eq. (9), the independent solutions of the associated Eq. (3) are

$$y_1(z) = (\phi\xi)^{1/2} H^{(1)}_{2/3}(\xi)\, e^{7\pi i/12}, \tag{10}$$

$$y_2(z) = (\phi\xi)^{1/2} H^{(2)}_{2/3}(\xi)\, e^{-7\pi i/12}. \tag{11}$$

Because of the stated behavior of $p(z)$ at $z = z_0$, Ψ' will vanish at $z = z_0$, whereas Ψ will be finite. This implies then that $q_1(z)$ will be nonsingular at $z = z_0$. From the behavior of $p(z)$ for $z \to \infty$, it can be shown that

$$\int_z^\infty p^{1/2} q_1(z)\, dz < \infty$$

for $z_0 < z \leqslant \infty$, and that $\Phi(z) \to \infty$ as $z \to \infty$.

The asymptotic behavior of y_1 and y_2, which will be needed later, is presented as follows. For $z \to \infty$, (with s fixed), or for $|s| \to \infty$ (with z fixed and $z > z_0$), y_1 and y_2 have the asymptotic form

$$y_1(z) \sim \left(\frac{2\phi}{\pi}\right)^{1/2} e^{i\xi} \tag{12}$$

$$y_2(z) \sim \left(\frac{2\phi}{\pi}\right)^{1/2} e^{-i\xi} \tag{13}$$

provided that $-\pi < \arg s < \pi$. For $|s| \to \infty$, with z fixed and $z < z_0$, the asymptotic behavior is given by the relations

$$y_1(z) \sim \left(\frac{2\phi}{\pi}\right)^{1/2} \{e^{i\tilde{\xi}} - ie^{-i\tilde{\xi}}\} \tag{14}$$

$$y_2(z) \sim -i \left(\frac{2\phi}{\pi}\right)^{1/2} e^{i\tilde{\xi}}, \tag{15}$$

where

$$\tilde{\xi} = e^{-2\pi i}\xi = -i\,|\,\Phi\,|\, s,$$

provided that

$$-\frac{\pi}{2} < \arg s < \frac{3\pi}{2}.$$

3. The Discrete Spectrum

The solution of Eq. (1) satisfying the following boundary conditions at $z = 0$

$$y(0) = \sin\alpha, \qquad y'(0) = -\cos\alpha \tag{16}$$

is given by

$$y(z) = \cos \alpha K(z, 0) + \sin \alpha K_x(z, 0) + \int_0^z K(z, x)\, \theta(x)\, y(x)\, dx, \tag{17}$$

where

$$K(z, x) = -\frac{\pi i}{4s} [y_1(z)\, y_2(x) - y_2(z)\, y_1(x)] \tag{18}$$

and

$$\theta(x) = [q(x) - q_1(x)]. \tag{19}$$

In the above equation, $K_x(z, 0)$ denotes the value of the partial derivative of $K(z, x)$ with respect to x, at $x = 0$.

In considering the discrete spectrum, interest will be limited to real negative values of λ, hence, set

$$s = it \tag{20}$$

where t is real and positive. As a preliminary, bounds upon the solution $y(z, t)$ will be found for arbitrary t. The functions $\eta(z)$ and $\nu(z)$ will be introduced, where $\eta(z)$ will be precisely specified by the relation

$$\eta(z) = \begin{cases} t \mid \Phi \mid & \text{for} \quad z \geqslant z_0 \\ 0 & \text{for} \quad z \leqslant z_0 \end{cases} \tag{21}$$

and $\nu(z)$ will be unspecified, except that it be bounded and positive definite in the interval $0 \leqslant z < \infty$ and have the following asymptotic behavior for $z \to \infty$

$$\nu(z) \sim \phi^{1/2}(z). \tag{22}$$

A particular choice for $\nu(z)$ could be given by the relations

$$\begin{aligned} \nu(z) &= \phi^{1/2}(z_1) \qquad z \leqslant z_1 \\ \nu(z) &= \phi^{1/2}(z) \qquad z \geqslant z_1\,, \end{aligned}$$

where z_1 is some arbitrary point lying in the open interval $(z_0\,, \infty)$. From the asymptotic behavior of the functions $y_1(z)$ and $y_2(z)$, it is seen that

$$\left| K(z, x) \frac{\exp[\eta(x) - \eta(z)]}{\nu(x)\, \nu(z)} \right| \qquad x \leqslant z,$$

and

$$\left| K_x(z, 0) \frac{\exp \eta(z)}{[t\nu(z)]} \right|$$

are uniformly bounded with respect to z and t, for $0 < \rho \leqslant t \leqslant \infty$. Hence, on setting

$$w(z) = e^{-\eta(z)} \frac{y(z)}{\nu(z)} \tag{23}$$

one obtains from Eq. (17) the inequality

$$| w | \leqslant C_1 + \frac{C_2}{t} + \left(\frac{C_3}{t}\right) \int_0^z | \theta(x)\, \nu^{-2}(x)| \, | w(x)| \, dx. \tag{24}$$

Since $\theta(z)\, \nu^{-2}(z)$ is nonsingular and bounded in $0 \leqslant z < \infty$, and has the asymptotic behaviour for $z \to \infty$

$$\theta(z)\, \nu^{-2}(z) \sim p^{1/2}(z)\, [q(z) - q_1(z)],$$

it can be shown from the a priori requirements upon p and q given in the introduction, that

$$\int_0^z | \theta \nu^{-2} | \, dx < \infty$$

for all z in $(0, \infty)$. It then follows [5] that

$$| w | < \left(C_1 + \frac{C_2}{t}\right) \left\{\exp\left[\frac{C_3}{t} \int_0^z | \theta \nu^{-2} | \, dx\right] - 1\right\}. \tag{25}$$

This implies that w will be bounded for all z and $0 < \rho \leqslant t < \infty$. From this, the asymptotic behaviour of Eq. (17) for $z \to \infty$, and arbitrary positive values of t, can be obtained as follows:

$$y(z) \sim \phi^{1/2}(z)\, e^{+\eta(z)} M(it) + \phi^{1/2}(z) \int_0^\infty L(x, z)\, \theta(x)\, y(x)\, dx + O(e^{-\eta(z)}), \tag{26}$$

where $M(it)$ is a real function given by the relation

$$M(it) = \frac{(2\pi)^{1/2}}{4t} \left[\cos \alpha y_1(0) + \sin \alpha y_1'(0) + \int_0^\infty y_1(x)\, \theta(x)\, y(x)\, dx\right] \tag{27}$$

and

$$L(x, t) = -\frac{(2\pi)^{1/2}}{4t} \begin{cases} e^{-\eta(z)} y_2(x) & x < z \\ e^{\eta(z)} y_1(x) & x > z. \end{cases}$$

Since $y(z)\, e^{-\eta(z)} \nu^{-1}(z)$ is bounded, it can be shown that for $z \to \infty$

$$\int_z^\infty L(x, z)\, \theta(x)\, y(x)\, dx \sim O\left\{\phi^{1/2}(z)\, e^{\eta(z)} \int_z^\infty | \theta\phi | \, dx\right\} = o\{\phi^{1/2}(z)\, e^{\eta(z)}\},$$

and

$$\int_0^z L(x, z)\, \theta(x)\, y(x)\, dx \sim O\left\{\phi^{1/2}(z) \int_0^{z^*} |\,\theta(x)|\, dx\right\}$$

$$+ O\left\{\phi^{1/2}(z)\, e^{\eta(z)} \int_{z^*}^z |\,\theta(x)\, \phi(x)|\, dx\right\}$$

$$\sim o\{\phi^{1/2}(z)\, e^{\eta(z)}\},$$

where z^* is chosen so that

$$\Phi(z) = 2\Phi(z^*).$$

Relationship (26) can be expressed in the form

$$y(z) \sim \phi^{1/2}(z)\, e^{\eta(z)}\{M(it) + O(1)\},$$

from which it is seen, that in order for $y(z)$ to belong to $L^2(0, \infty)$, $M(it)$ must vanish, i.e.,

$$\frac{1}{t} \cos \alpha y_1(0) + \frac{1}{t} \sin \alpha y_1'(0) + \frac{1}{t} \int_0^\infty y_1(x)\, \theta(x)\, y(x)\, dx = 0. \tag{28}$$

Since it can be shown that $M(s)$ is an analytic function of s regular for $\operatorname{Im} s > 0$, the zeros of $M(it)$ are isolated points. Hence, there is a point spectrum in $-\infty \leqslant \lambda < \infty$ given by the zeros of Eq. (28).

4. Asymptotic Behavior of the Point Spectrum

From Equation (25) it is seen that

$$|\, y(z)| \lesssim e^{\eta(z)} \nu(z)\, O\left(\frac{1}{t}\right)$$

for $t \to \infty$, from which it follows that the asymptotic behavior of Eq. (28) for $t \to \infty$, is given by

$$\frac{\cos \alpha}{t} y_1(0) + \frac{\sin \alpha}{t} y_1'(0) + O\left(\frac{1}{t^2}\right) = 0. \tag{29}$$

With the employment of the asymptotic expressions developed for $y_1(z)$ in Section 2, this can be expressed in the explicit form

$$\sin\left[t\,|\,\Phi(0)| + \frac{\pi}{4}\right] + O\left(\frac{1}{t}\right) = 0 \qquad \sin \alpha \neq 0$$

and

$$\cos\left[t\,|\,\Phi(0)| + \frac{\pi}{4}\right] + O\left(\frac{1}{t}\right) = 0 \qquad \sin\alpha = 0.$$

The asymptotic values of the point spectrum λ_N, where N takes on positive integer values, are immediately obtainable and are given by

$$\lambda_N = -\,t_N{}^2, \tag{30}$$

$$t_N = \frac{\pi(N - \frac{1}{4})}{|\,\Phi(0)|} + O\left(\frac{1}{N}\right) \qquad \sin\alpha \neq 0, \tag{31}$$

$$t_N = \frac{\pi(N + \frac{1}{4})}{|\,\Phi(0)|} + O\left(\frac{1}{N}\right) \qquad \sin\alpha = 0. \tag{32}$$

It is seen that the main effect of the pole upon the spectrum, is to produce a point spectrum which is unbounded below.

5. Some Special Considerations

For some special cases, uniform asymptotic solutions of Eq. (1) may be obtained, with the result that error bounds may be established for the asymptotic values of the discrete spectrum. By performing the transformation

$$\zeta = (\tfrac{3}{2}\,\Phi)^{2/3},$$

where Φ is defined by Eq. (7) and setting

$$y = \left(\frac{\zeta}{p}\right)^{1/4} W,$$

one can reduce Eq. (1) to the form

$$\frac{d^2W}{d\zeta^2} = W\left[-\lambda\zeta + \frac{3}{4}\,\frac{1}{\zeta^2} + f(\zeta)\right], \tag{33}$$

where

$$f(\zeta) = \zeta q(z) + \frac{1}{4}\,\frac{(\zeta p)''}{\zeta p} - \frac{3}{16}\left[\frac{(\zeta p)'}{\zeta p}\right]^2 - \frac{1}{4}\,\frac{(\zeta p)'}{\zeta^2 p},$$

and the prime represents differentiation with respect to ζ. Uniform asymptotic solutions for large λ can be obtained from Olver [6], provided $f(\zeta)$ can be represented in the form

$$f(\zeta) = \zeta h(\zeta^3), \tag{34}$$

where $h(\zeta^3)$ is an analytic function of ζ^3. In this case (33) reduces to Olver's case D.

Representation (34) is possible if $q(z)$ is an analytic function of ζ^3 and $p(z)$ is expressible in the form $\zeta p = \exp H(\zeta^3)$, where $H(\zeta^3)$ is an analytic function of ζ^3.

References

1. V. H. Weston, *Can. J. Phys.* **40** (1962), 431.
2. V. H. Weston, *Can. J. Phys.* **40** (1962), 446.
3. E. C. Titchmarsh, "Eigenfunction Expansions," 2nd ed., p. 116–119, Oxford Univ. Press (Clarendon), London, 1962.
4. R. E. Langer, *Amer. Math. Soc.* **37** (1935), 397–416.
5. E. C. Titchmarsh, "Eigenfunction Expansions," 2nd ed., p. 115, Lemma 5.6, Oxford Univ. Press (Clarendon), London, 1962.
6. F. W. J. Olver, *Philos. Trans. Roy. Soc. London* **249A** (1956), 65–97.